Exact Solutions of Simple Nonlinear Difference Equation Systems that show Chaotic Behavior

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Exact solutions are given for May's simple difference equation and the broken linear model, both for extreme values of the parameters, and the relation between these two systems for the solvable cases is clarified. The invariant measure and the correlation function for each case are calculated using the exact solutions. The initial-value dependence of the behavior of the solutions, choatic or periodic, is completely determined. By using the exact solution, encounters of any two nonperiodic orbits can be precisely analyzed.

1. Introduction

May, in his well-known article [1], studied one of the simplest nonlinear difference equation of ecological interest, namely,

$$X_{n+1} = a X_n (1 - X_n), \qquad (1)$$

where X_n represents the normalized ($0 \leqq X_n \leqq 1$) population of the n^{th} generation of a species under consideration and a is a positive constant with restriction $0 \leqq a \leqq 4$ which is necessary to keep the population of each generation normalized. Equation (1) can be regarded as one of the discrete versions of the logistic equation. The striking result is that (1) has solutions that show chaotic behavior for larger values of a. Specifically, for $a > 3.8284\ldots$ the solution of period p of any integral value exists as well as chaotic solutions. The fact that a simple equation such as (1) can produce chaotic solutions stimulated mathematicians and physicists to pursue further studies of the nature of chaos [2]–[5].

In certain circles it it well-known that some nonlinear difference equations that show random behavior can be solved exactly. For instance, Bender and Orszag in their textbook [6] studied extensively a difference equation which results from an attempt to compute the square root of -3 utilizing Newton's iteration method, namely, $a_{n+1} = (1/2)(a_n - 3/a_n)$. They showed that the exact solution of this equation can be written as $a_n = \sqrt{3} \cot 2^n \theta_0$.

Using similar techniques, we solve (1) for $a = 4$ and analyze in detail the dependence of the nature of the solution on the initial conditions. We also present the exact solution of the so-called broken linear model that is defined as

$$X_{n+1} = \begin{cases} \alpha X_n & (0 \leqq X_n \leqq 1/2), \\ \alpha(1 - X_n) & (1/2 \leqq X_n \leqq 1), \end{cases} \qquad (2)$$

for $\alpha = 2$.

The exact solution of (1) for $a = 4$ can also be inferred via the method of conjugation presented by Grossmann and Thomae [2] and has been used by some researchers [7].

2. Exact Solutions

We first show that (1) for $a = 2$ can also be exactly solved. Using the transformation

$$X_n = (1/2)(1 - Y_n) \qquad (3)$$

in (1) we have

$$Y_{n+1} = (a/2)\, Y_n^2 + (1 - a/2) \qquad (4)$$

which reduces to

$$Y_{n+1} = Y_n^2 \qquad (5)$$

for $a = 2$. Hence, we have

$$Y_n = Y_0^{2^n}. \qquad (6)$$

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Thus the exact solution of (1) for $a = 2$ is

$$X_n = (1/2)[1 - (1 - 2X_0)^{2^n}]. \tag{7}$$

It is clear, for any $X_0\,(0 \leqq X_0 \leqq 1)$, X_n converges to 1/2 for large n in a monotonic fashion.

For $a = 4$ we substitute

$$Y_n = \cos \pi \xi_n \tag{8}$$

into (4). Then we have

$$\begin{aligned}\cos \pi \xi_{n+1} &= 2\cos^2 \pi \xi_n - 1\\ &= \cos 2\pi \xi_n\\ &= 2\cos^2 2\pi \xi_{n-1} - 1\\ &= \cos 2^2 \pi \xi_{n-1}.\end{aligned} \tag{9}$$

Hence, repeating this procedure, we have

$$\cos \pi \xi_n = \cos 2^n \pi \xi_0. \tag{10}$$

Now impose a restriction on ξ_n such that $0 \leqq \xi_n \leqq 1$ for any n. Then we have from (9)

$$\xi_{n+1} = \begin{cases} 2\xi_n & (0 \leqq \xi_n \leqq 1/2),\\ 2(1-\xi_n) & (1/2 \leqq \xi_n \leqq 1).\end{cases} \tag{11}$$

This is the broken linear model (Eq. (2)) for $\alpha = 2$. Since

$$\xi_0 = (1/\pi)\cos^{-1} Y_0, \tag{12}$$

where $0 \leqq \cos^{-1} Y \leqq \pi$ is the principal value of arccosine function, we obtain the exact solution of (1) for $a = 4$ as, from (10),

$$\begin{aligned}X_n &= (1/2)(1 - \cos \pi \xi_n) && (13)\\ &= (1/2)(1 - \cos 2^n \pi \xi_0)\\ &= (1/2)[1 - \cos\{2^n \cos^{-1}(1 - 2X_0)\}], && (14)\end{aligned}$$

which can be rewritten in a simpler form as

$$X_n = \sin^2(2^n \cdot \sin^{-1}\sqrt{X_0}). \tag{15}$$

On the other hand, we have, from (10)

$$\xi_n = (1/\pi)\cos^{-1}\cos(2^n \pi \xi_0). \tag{16}$$

This is the exact solution of (11).

The system (11) has a uniform invariant density (distribution function) over $0 \leqq \xi \leqq 1$, as shown in [2] and [3]. Hence the invariant measure for X is determined by

$$W(X)\,\mathrm{d}X = \mathrm{d}\xi, \tag{17}$$

where $W(X)$ is the invariant density of the solution for X. Differentiating both sides of (13) with respect to ξ_n, we have (by dropping the subscript n)

$$W(X) = \mathrm{d}\xi/\mathrm{d}X = \frac{1}{\pi\sqrt{X(1-X)}}, \tag{18}$$

which is the invariant density for the system expressed by (1) for $a = 4$ [2].

Once we have the invariant measure we can calculate the correlation function which is defined as

$$C(n) \equiv \frac{\langle (X_n - \langle X_n \rangle)(X_0 - \langle X_0 \rangle)\rangle}{\langle (X_0 - \langle X_0 \rangle)^2 \rangle}, \tag{19}$$

where

$$\langle \cdots \rangle \equiv \int_0^1 W(X)(\cdots)\,\mathrm{d}X, \tag{20}$$

yielding the results,

$$C(0) = 1, \quad C(n) = 0 \quad (n \geqq 1). \tag{21}$$

The broken linear model (11) has a uniform invariant density, as already mentioned, and has the same correlation function as (21) [2].

3. Fixed Points

Since we now have the exact solutions of (1) for $a = 4$, and (11), we can readily locate the fixed points of period p, which we denote as $X^{(p)}$ for any value of p. By using (13), $X^{(p)}$ can be transformed into the ξ-space as

$$X^{(p)} = (1/2)(1 - \cos \pi \xi^{(p)}). \tag{22}$$

If the exact solution (13) has the fixed point of period p, then the relation in ξ-space

$$\cos(2^p \pi \xi^{(p)}) = \cos \pi \xi^{(p)} \tag{23}$$

must be satisfied. Furthermore, the condition

$$\cos(2^q \pi \xi^{(p)}) \neq \cos \pi \xi^{(p)} \tag{24}$$

for any positive integer $q < p$, must be fulfilled. Therefore, the fixed points of period p are given by

$$\xi^{(p)} = 2k/(2^p \pm 1), \tag{25}$$

where

$$k = 0, 1, 2, \ldots, 2^{p-1} - 1/2 \pm 1/2,$$

with the condition

$$\xi^{(p)} \neq 2k'/(2^q \pm 1), \tag{26}$$

where

$$k' = 0, 1, 2, \ldots, 2^{q-1} - 1/2 \pm 1/2$$

for every integer q that satisfies $q < p$.

In the following we list the explicit expressions of the fixed points for small p values. The fixed points of period 1 are given by (25) as

$$\xi^{(1)} = 0 \tag{27}$$

and

$$\xi^{(1)} = 2/3 . \tag{28}$$

In the original X-space these correspond to

$$X^{(1)} = 0 \tag{29}$$

and

$$X^{(1)} = 3/4 . \tag{30}$$

The fixed point of period 2 is found to be the pair

$$\xi^{(2)} = 2/5 \quad \text{and} \quad 4/5 . \tag{31}$$

In the original space this pair becomes

$$X^{(2)} = (5 - \sqrt{5})/8 \quad \text{and} \quad (5 + \sqrt{5})/8 . \tag{32}$$

The fixed points of period 3, which are crucial for emergence of chaos according to the theorem of Li and Yorke [8], are two sets of fixed points

$$\xi^{(3)} = 2/7, 4/7, \text{and } 6/7 \tag{33}$$

and

$$\xi^{(3)} = 2/9, 4/9, \text{and } 8/9 . \tag{34}$$

Similarly, we find that there are 3 sets of fixed points of period 4 and 6 sets of fixed points of period 5, and so on.

Finally, in this section, we remark that the fixed point of period p which is expressed by the form with the plus sign in the denominator of (25), can be transformed into the same expression with the minus sign and the doubled period. This remark is necessary for the consideration given in the next section. We denote the fixed point of period p given by (25) with the plus (minus) sign as $\xi^{(p)+}$ ($\xi^{(p)-}$). Then $\xi^{(p)+}$ can be written as

$$\xi^{(p)+} = \frac{2k}{2^p + 1} = \frac{2k''}{2^{2p} - 1} , \tag{35}$$

where

$$k'' \equiv k(2^p - 1) . \tag{36}$$

Comparing the right-hand side of (35) with (25), we see that $\xi^{(p)+}$ is expressed in the form of the fixed point of period $2p$ with the minus sign, i.e., $\xi^{(2p)-}$. The essential point is that all the fixed points are written in the form of $\xi^{(p)-}$.

4. Periodic and Nonperiodic Solutions

In this section we prove that the exact solution of (1) for $a = 4$ with the condition (10) becomes nonperiodic if the initial value in the ξ-space is an irrational number, and periodic if it is any rational number.

If we consider an expression

$$\cos(2^j \pi \xi_0) = \cos(\pi \xi^{(p)}) \tag{37}$$

which locates the initial value ξ_0 that gives a solution that falls into a fixed point of period p after j iterations, we see that the equation cannot hold for irrational ξ_0's because from (25) it is self-evident that $\xi^{(p)}$ is a rational number for any p. Therefore, it can be concluded that the solution with the initial value ξ_0 that is an irrational number is nonperiodic.

Now we consider the case in which the initial value ξ_0 is a rational number, which we denote as l/m, where $l < m$ and l and m have no common divisors. In the following we prove that any rational number, l/m where l is even and m odd, is a periodic fixed point of a certain period, and the exact solution starting from any initial value of the form $\xi_0 = l/m$ where both l and m are odd or l is odd with m even eventually falls into one of the periodic fixed points.

First we remark, from (25), (26), and (35), that any initial value that can be expressed in the form

$$\xi_0 = 2k/(2^p - 1) , \tag{38}$$

where

$$k = 0, 1, 2, \ldots, 2^{p-1} - 1 ,$$

can take the same value at the p iterations, but it sometimes has a smaller period q ($q < p$). The smallest of such q is called an actual period. This is a divisor of p.

Case 1: l(even)/m(odd)

Here we first refer to Euler's theorem in number theory [9] which is crucial to our discussion: let r and m be relatively prime integers, then we have

$$r^{\Phi(m)} = 1 \quad (\text{mod } m) , \tag{39}$$

where $\Phi(m)$ is the Euler's function which is defined as the number of integers among $1, 2, \ldots, m - 1$ that are relatively prime to m (if m is a prime number, then $\Phi(m) = m - 1$ and $\Phi(1) = 1$). For case 1, ξ_0 can be expressed as $\xi_0 = 2l'/m$. Since m is odd, we can take $r = 2$. Then from Euler's theorem stated above, we can conclude that $(2^{\Phi(m)} - 1)/m$ is an integer. Hence, we can rewrite ξ_0 as

$$\xi_0 = 2k/(2^{\Phi(m)} - 1)\,. \tag{40}$$

Therefore, any rational number of case 1 is the fixed point of period $\Phi(m)$, and possibly the actual period ($q \leqq \Phi(m)$) is a divisor of $\Phi(m)$. (In the Appendix we show how to find the actual period of the periodic fixed point.)

Case 2: l(odd)/m(odd)

The point ξ_1 obtained by the first iteration becomes either $2\xi_0 = 2l/m$ for ($l/m \leqq 1/2$) or $2(1 - \xi_0) = 2l'/m \leqq 1$ where $l' = m - l$ for $1/2 \leqq l/m \leqq 1$. Therefore, ξ_0 for case 2 falls into ξ_1 which is a periodic fixed point.

Case 3: l(odd)/m(even)

We put $m = 2^s m'$ where s is an integer and m' an odd number. Then we define l' by the relation

$$\cos^{-1}\cos(2^s \pi\, \xi_0) = (l'/m')\,\pi \quad (0 \leqq l'/m' \leqq 1)\,. \tag{41}$$

Therefore, after s iterations the rational number of case 3 reduces to a rational number of either case 1 or 2, already discussed above. Therefore, the rational number of case 3 eventually falls into a periodic fixed point.

5. Encounters of Two Orbits

According to Li and Yorke [8], two nonperiodic orbits repeatedly come close and then separate each other. As we have the exact solution for the systems (1) for $a = 4$ and (2) for $\alpha = 2$, we can predict when these encounters take place as precisely as we wish provided the initial conditions of the two orbits are given. Here we discuss the close encounters of two orbits using the exact solution for May's equation for $a = 4$ (Equation (13)).

Let two different initial values be X_0 and X'_0 which, from (13), can be expressed as

$$X_0 = (1/2)\,(1 - \cos \pi\, \theta_0) \tag{42}$$

and

$$X'_0 = (1/2)\,(1 - \cos \pi\, \psi_0)\,, \tag{43}$$

where $0 < \theta_0, \psi_0 < 1$. We then have

$$X_n = (1/2)\,(1 - \cos \pi\, 2^n \theta_0)\,, \tag{44}$$

$$X'_n = (1/2)\,(1 - \cos \pi\, 2^n \psi_0) \tag{45}$$

and the difference between these two orbits, namely,

$$\Delta_n = X_n - X'_n = \sin \pi\, 2^n \omega_+ \sin \pi\, 2^n \omega_-\,, \tag{46}$$

where

$$\omega_\pm \equiv (1/2)\,(\theta_0 \pm \psi_0)\,. \tag{47}$$

In order to have a close encounter of the two orbits such that $|\Delta_n| < \varepsilon$, where ε is an arbitrary small number, it is sufficient to make either of the two sine functions on the right-hand side of (46) smaller than ε. Now let us define

$$\{2^n \omega\} \equiv (1/\pi) \sin^{-1}(\sin 2^n \pi\, \omega)\,, \tag{48}$$

where $\sin^{-1}$ denotes the principal value of the arcsine function. Then we have

$$-(1/2) \leqq \{2^n \omega\} \leqq 1/2 \tag{49}$$

and the relation $|\Delta_n| < \varepsilon$ turn out to be either

$$|\{2^n \omega_+\}| < \varepsilon/\pi \equiv \varepsilon' \tag{50}$$

or

$$|\{2^n \omega_-\}| < \varepsilon/\pi \equiv \varepsilon'\,. \tag{51}$$

Putting $\varepsilon' = 2^{-m}$, where m is a positive integer, we can determine n the number of iterations successively as $n_1^+ < n_2^+ < \cdots$ and $n_1^- < n_2^- < \cdots$ that satisfy (50) and (51), respectively. If we represent ω_+ (or ω_-) in the binary system, we see that (50) (or (51)) is satisfied each time when we have m or more successive 0's or 1's appear in the representation. As an example, we take

$$\omega_- = 0.0\,1\,0\,1\,0\,1\,1\,1\,0\,1\,0\,0\,0\,1\,1\,\ldots\,,$$
$$\omega_+ = 0.1\,0\,0\,0\,1\,1\,0\,1\,1\,1\,1\,0\,0\,0\,1\,\ldots\,, \tag{52}$$

and $m = 3$. Then we have

$$n_1^- = 5\,, \quad n_2^- = 10\,,$$
$$n_1^+ = 1\,, \quad n_2^+ = 7\,, \quad n_3^+ = 8\,, \quad n_4^+ = 11\,. \tag{53}$$

see that at the number of iterations

$$n = 1, 5, 7, 8, 10, 11, \ldots \tag{54}$$

the two orbits selected here come close within the distance $\pi/2^3$.

A similar analysis can be made for the largest separation between any two orbits.

Appendix

The Actual Period of the Periodic Solution

In this appendix we show how to find the actual period of the periodic solution of (17 for $a = 4$ when we are given such a solution (i.e., any even/odd fraction). By actual period we mean the smallest number of repetition of iterations for the return to the initial point (see (38)). For this purpose we first introduce a quantity called the exponent [8]. The relation (see (39))

$$r^{\Phi(m)} = 1 \quad (\text{mod } m) \tag{A1}$$

implies that there exists a smallest positive integer d such that

$$r^d = 1 \quad (\text{mod } m)\,. \tag{A2}$$

The integer d is called the exponent to which r belongs (mod m). The exponent d is obviously a divisor of $\Phi(m)$.

From (35) and (38) we know that any periodic solution of period p can be written as either of the following two,

$$\xi_0 = \begin{cases} \dfrac{2k}{2^p - 1} & (0 \leqq k \leqq 2^{p-1} - 1) \quad \text{(A3)} \\[2ex] \dfrac{2k}{2^p + 1} = \dfrac{2k'}{2^{2p} - 1} & (0 \leqq k \leqq 2^{p-1})\,, \quad \text{(A4)} \end{cases}$$

where p is the smallest integer that satisfies either of these expressions.

We also have $\xi_0 = l\,(\text{even})/m\,(\text{odd}) = 2\,l'/m$. When m is given, we can (taking $r = 2$), in principle, obtain $\Phi(m)$ and d. If d turns out to be odd, then from (A3) we have

$$\xi_0 = 2k/(2^d - 1)\,, \tag{A5}$$

and d is the period, because there is no integer smaller than d that can be written in the form of (A4), or that satisfies

$$2^d + 1 = 0 \quad (\text{mod } m)\,. \tag{A6}$$

If there exist integers smaller than d that satisfy (A6), we have a contradiction. Let d' be the smallest integer smaller than d that satisfies (A6). We then can write $d = j\,d' + t$, where $0 \leqq t < d' \leqq d$ and j integer. The relation $2^d - 1 = 0$ (mod m) yields

$$2^t + 1 = 0\ (\text{mod } m) \quad (\text{for odd } j) \tag{A7}$$

and

$$2^t - 1 = 0\ (\text{mod } m) \quad (\text{for even } j)\,. \tag{A8}$$

Equation (A7) contradicts the assumption that d' is the smallest integer that satisfies (A6), and (A8) contradicts the assumption that $d(>t)$ is the smallest integer that satisfy (A2) for $r = 2$. Therefore, d is the smallest integer that satisfies (A6). Hence, for odd d we have d as the period p. (See example 1 below.)

If d is even, let $d \equiv 2^S d'$ where d' is odd. Then d' can be written as $d' = d_1 d_2$, where d_1 and d_2 are odd integers including 1. We factorize $2^d - 1$ as

$$2^d - 1 = (2^{2^{S-1}d'} + 1)(2^{2^{S-2}d'} + 1) \cdots (2^{d'} + 1)(2^{d'} - 1)\,. \tag{A9}$$

Now $2^{2^h d''} + 1$ ($d'' = d_1$ or d_2) are divisors of $2^{2^h d'} + 1$. If m is a divisor of at least one of $2^{2^h d''} + 1$ ($h = 0, 1, \ldots, S-1$), then the actual period p is the smallest $2^h d''$ among the all possible pairs of d_1 and d_2 and all h satisfying $2^{2^h d''} + 1 = 0$ (mod m) (example 2). Otherwise, d is the period (example 3).

Example 1: $m = 7,\ \Phi(7) = 6,\ d = 3,\ p = 3,$

example 2: $m = 9,\ \Phi(9) = 6,\ d = 6,\ p = 3,$

$m = 17,\ \Phi(17) = 16,\ d = 8,\ p = 4,$

example 3: $m = 15,\ \Phi(15) = 8,\ d = 4,\ p = 4.$

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